CS 237: Probability in Computing

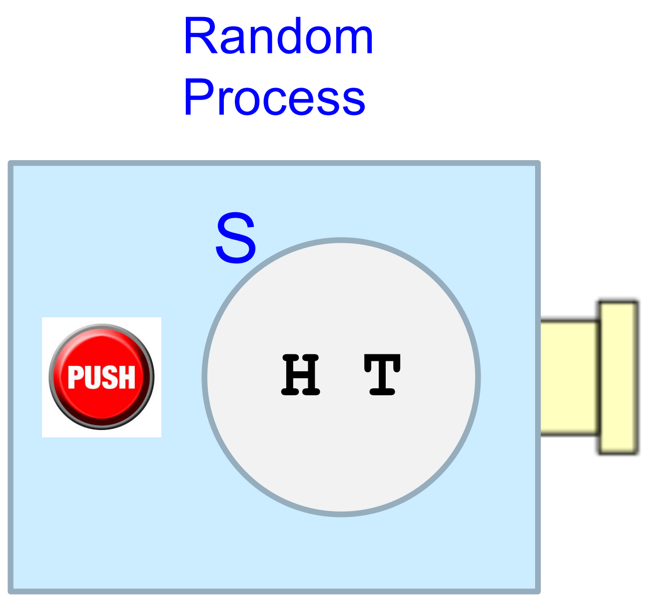
Wayne Snyder
Computer Science Department
Boston University

Lecture 10:

- Introduction to Random Variables

Random Experiments and Random Variables

A Random Experiment is a process that produces uncertain outcomes from a well-defined sample space.



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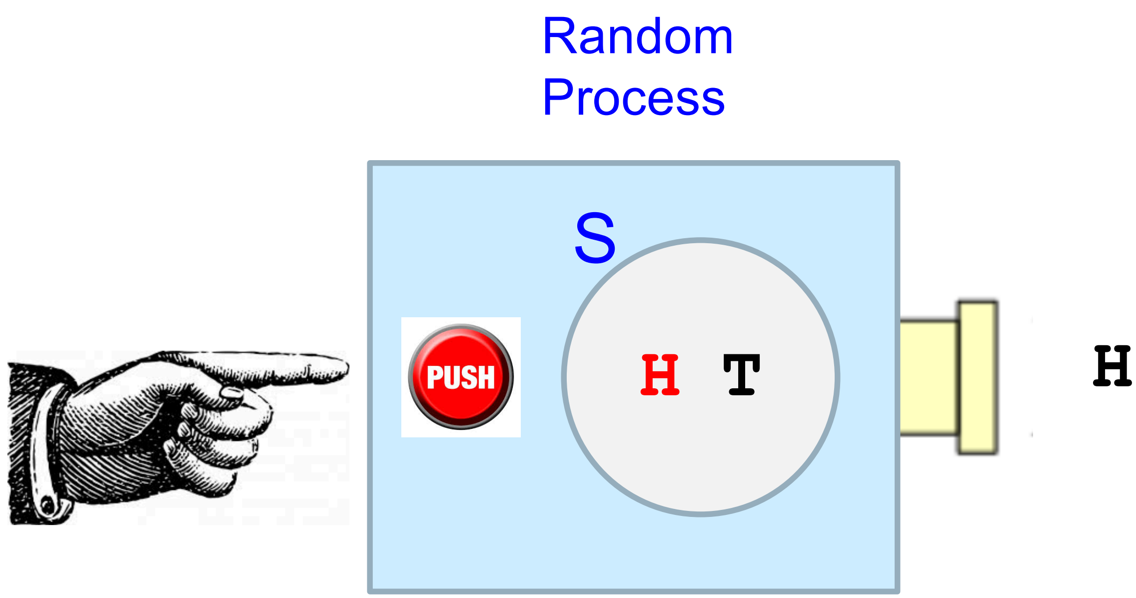


RandomVariables

In order to formalize this notion, the notion of a Random Variable has been developed. A Random Variable X is a function from a sample space S into the reals:

$$X : S \rightarrow \mathcal{R}$$

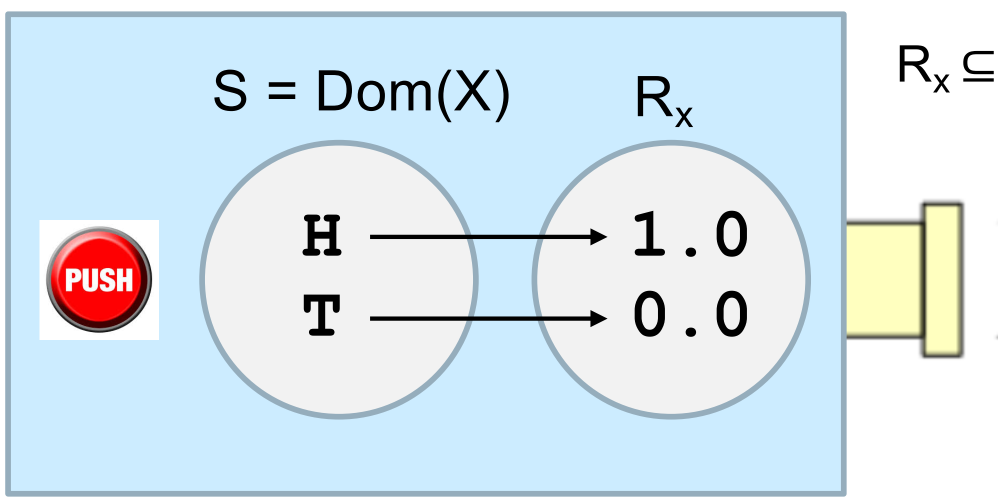
Now when an outcome is requested, the sample point is translated into a real number:

S = Domain(X)

X

$R_x =_{def} \text{Range}(X)$

$R_x \subseteq \mathcal{R}$



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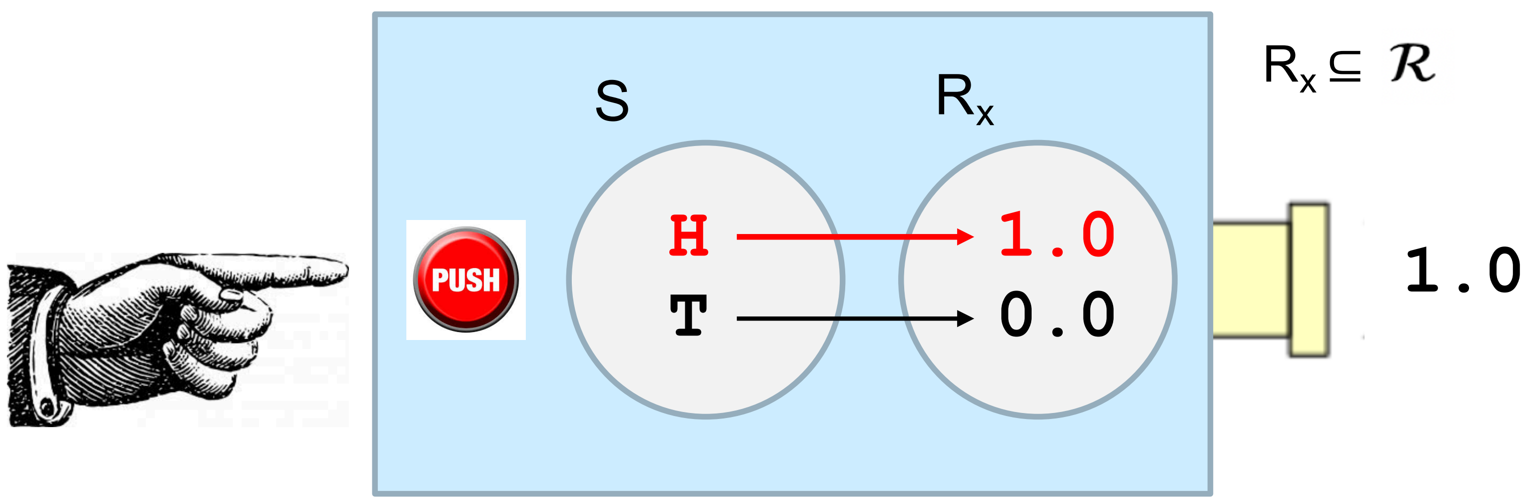
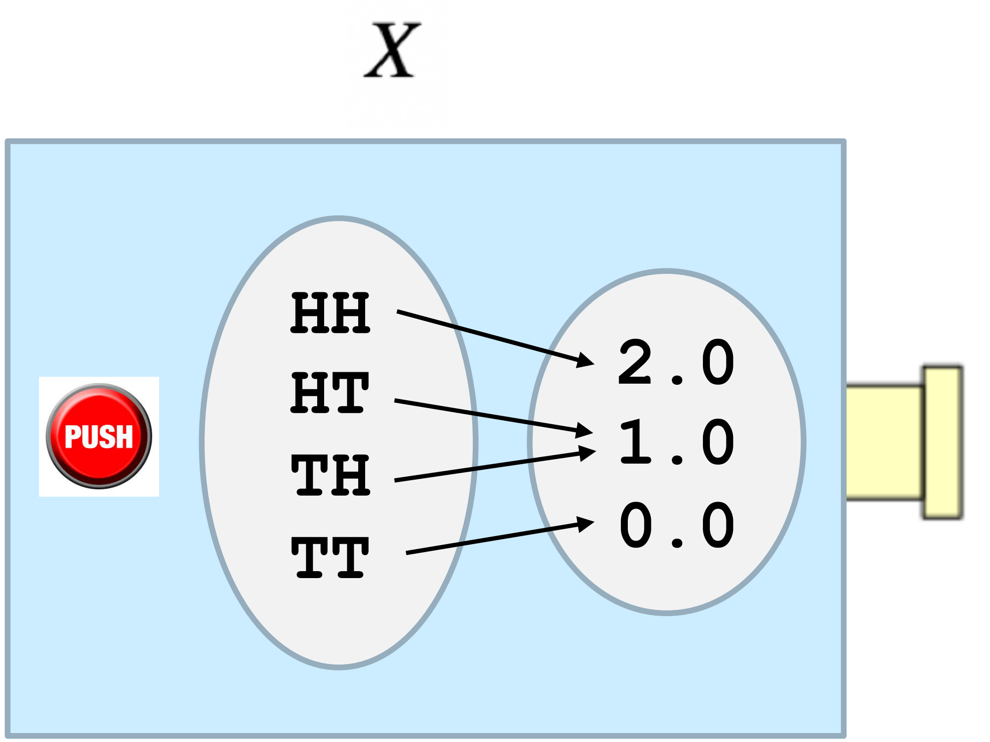


Random Variables

This may seem awkward, but it helps to explain the difference between random experiments whose literal outcomes are not numbers, but which are translated into numbers for clarity.

Example: X = "the number of heads which appear when two fair coins are flipped."



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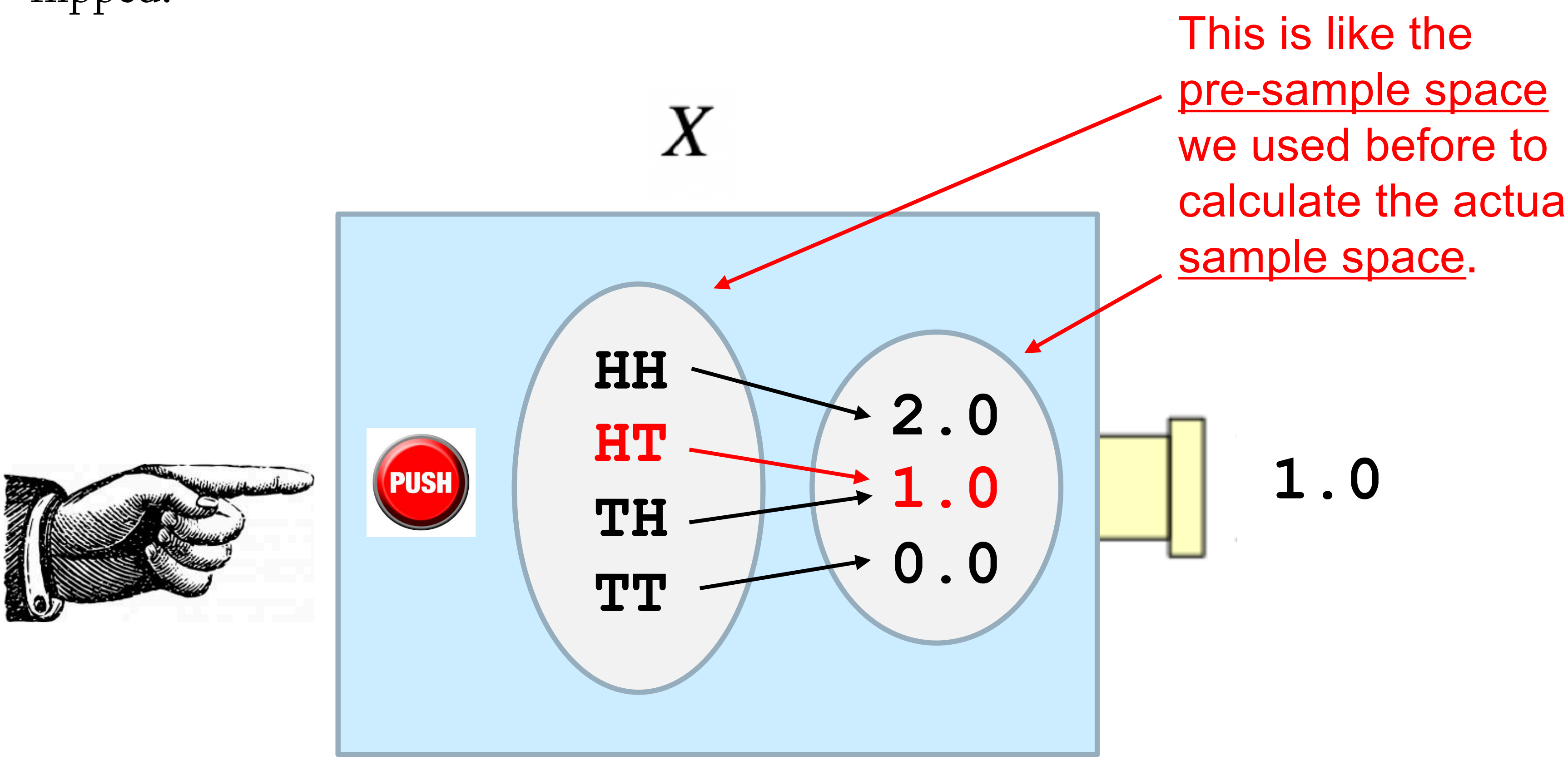
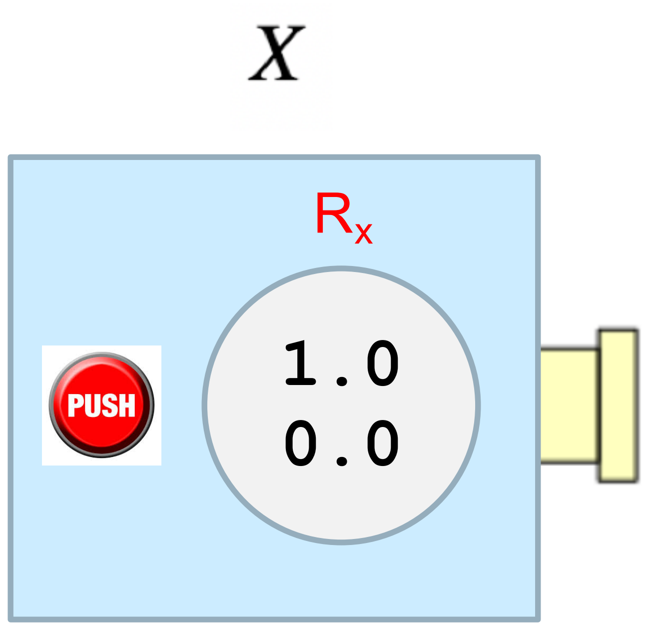


Random Variables

In general, in this class we will call the possible outputs R_x, since this the symbol used in your textbook, although you could just think of it as the sample space from which the outputs are drawn.



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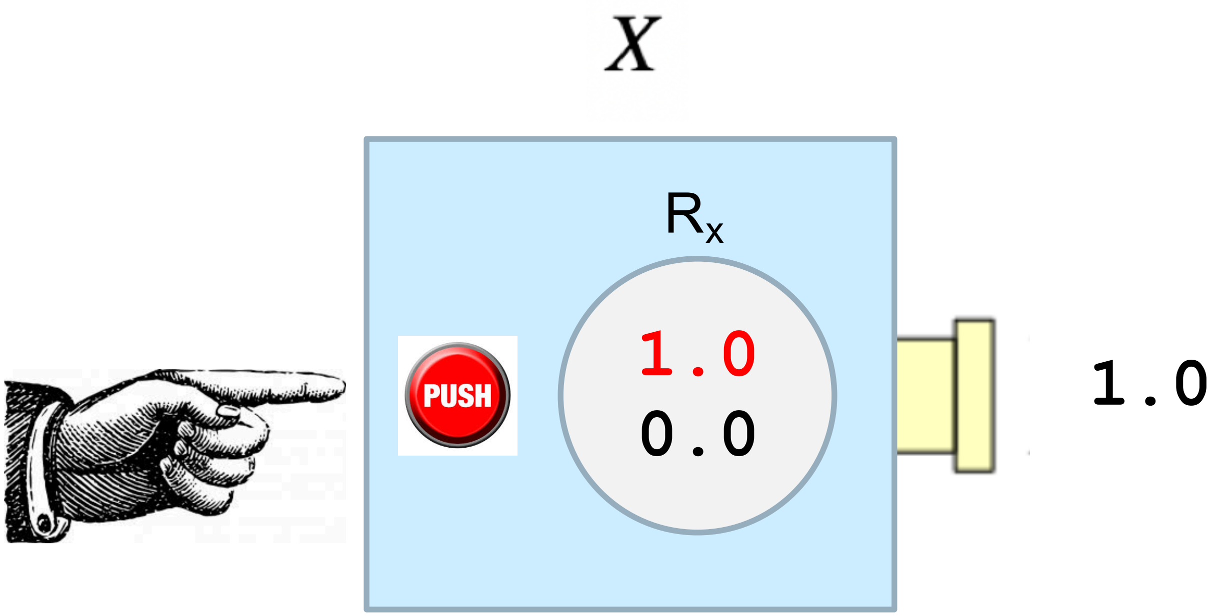


Discrete vs Continuous Random Variables

A random variable X is called discrete if R_x is finite or countably infinite:

Example of finite random variable:

X = "the number of dots showing after rolling two dice"

R_X = { 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 }

Example of countably infinite random variable:

Y = "the number of flips of a coin until a head appears"

R_Y = { 1, 2, 3, ... }

For several weeks we will only consider discrete random variables!

A random variable is called continuous if Rx is uncountable. Example:

Z = "the distance of a thrown dart from the center of a circular target of 1 meter radius"

R_Z = [0.0 .. 1.0)

Probability Mass Function of a Discerete RandomVariable

A random variable X is called discrete if R_x is finite or countably infinite;

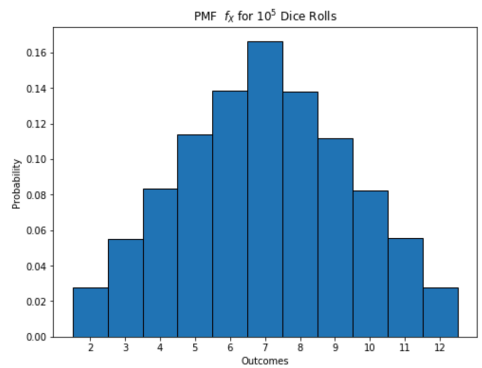
In this case the Probability Function will be given an new and improved name, it will be called the Probability Mass Function (PMF) and referred to by f_x:

Example of finite random variable:

X = "the number of dots showing after rolling two dice"

R_X = { 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 }

f_X = { 1/32, 2/32, 3/32, 4/32, 5/32, 6/32, 5/32, 4/32, 3/32, 2/32, 1/32 }



Discrete Random Variables: Probability Distributions

The Probability Mass Function (PMF) of a discrete random variable X is a function from the range of X into $\mathcal{R}$:

$$f_X : R_X \to [0..1]$$

such that (i) $\forall a \in R_x \;\; f_X(a) \geq 0$

(ii) $$\sum_{a \in R_x} f_X(a) = 1.0$$

If there is no possibility of confusion we will write f instead of f_X.

Discrete RandomVariables: Probability Distributions

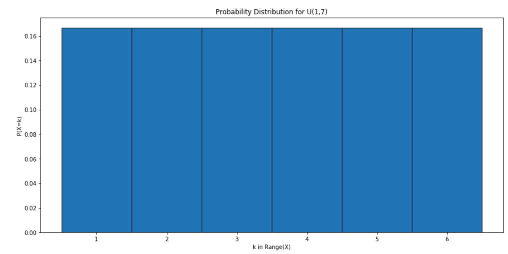
To specify a random variable precisely, you need to give the range R_X and the Probability Mass Function f_X.
Examples:

X = "The number of dots showing on a thrown die"

For simplicity, we simply list the values in Range(f_X) corresponding to the listing of R_X.

$$R_X = \{1, 2, 3, 4, 5, 6\}$$

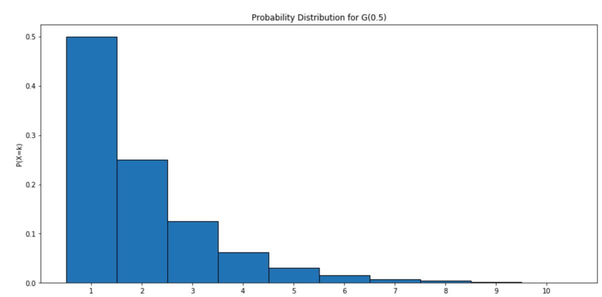
$$f_X = \{\ \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\ \}$$



Y = "The number of tosses of a fair coin until a head appears"

$$R_Y = \{\ 1, 2, 3,\ \ldots\ \}$$

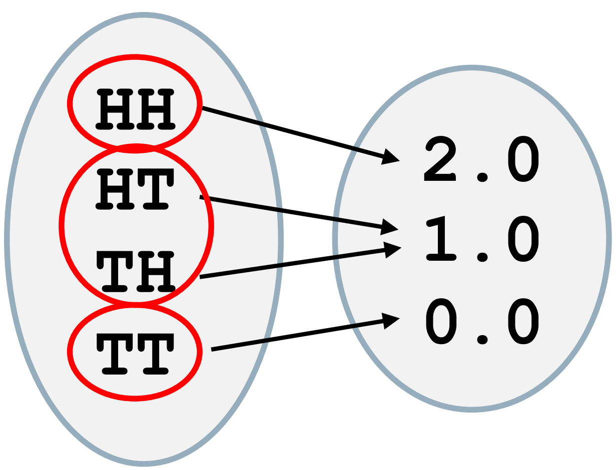
$$f_Y = \{\ \frac{1}{2}, \frac{1}{4}, \frac{1}{8},\ \ldots\}$$



Discrete RandomVariables: Probability Distributions

How does this relate to our first definition of a probability space, events, probability function, etc., etc. ??

Probability Space	Random Variable X
Sample Space	$\boldsymbol{R_X}$
Event	Subset of real numbers (with some restrictions).
Probability Function	Probability Distribution $\boldsymbol{f_X}$



For continuous random variables there are additional conditions about events having to be the countable sum of intervals on the real number line.

Discrete Random Variables: Probability Distributions

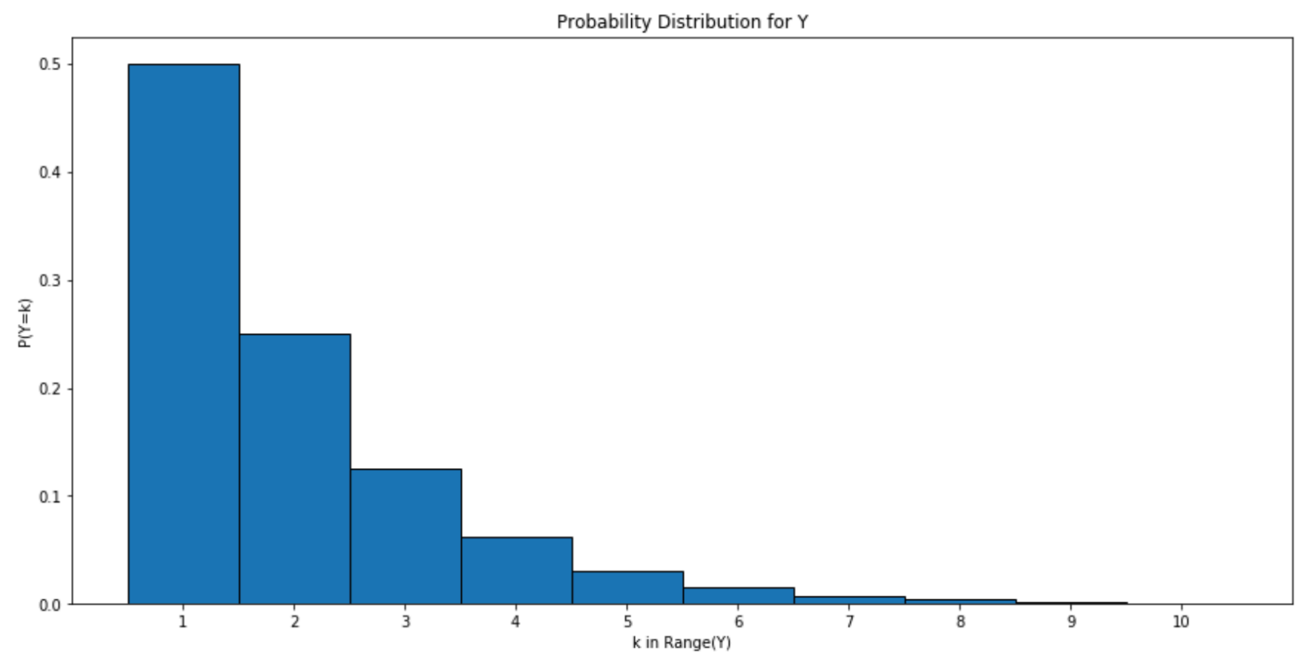
We will emphasize going forward the distributions of random variables, using graphical representations (as in HW 01) to help our intuitions.

Example:

Y = "The number of tosses of a fair coin until a head appears"

$$R_Y = \{\ 1, 2, 3,\ \ldots\ \}$$

$$f_Y = \{\ \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8},\ \ldots \}$$



Discrete RandomVariables: Notations

Notation:

$$P(X = k) \;=_{def}\; P_X(k)$$

$$P(X \neq k) \;=_{def}\; 1.0 - P_X(k)$$

$$P(X \leq k) \;=_{def}\; \sum_{a \leq k} P_X(a)$$

$$P(j \leq X \leq k) \;=_{def}\; \sum_{j \leq a \leq k} P_X(a)$$

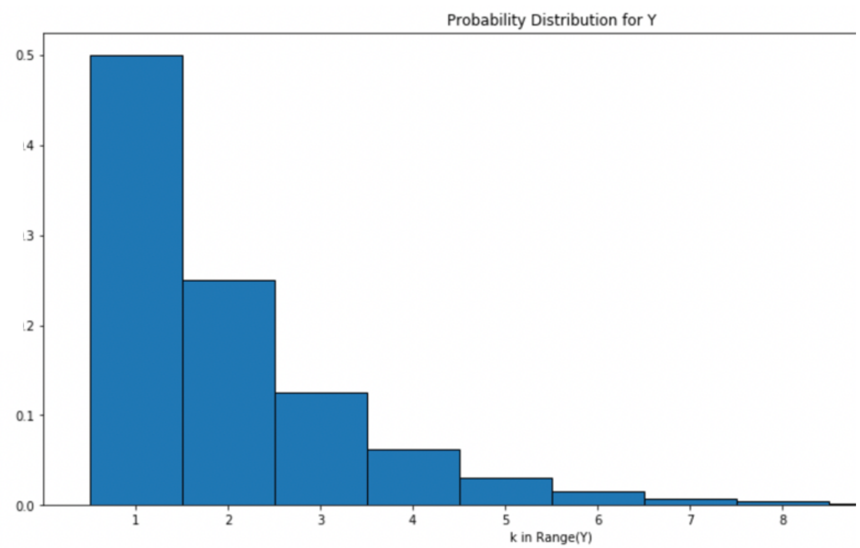
$$P(Y = 4) = \frac{1}{16}$$

$$P(Y < 4) = \frac{7}{8}$$

$$P(2 \leq Y \leq 4) = \frac{7}{16}$$

$$R_Y = \{ 1, 2, 3, \ldots \}$$

$$P_Y = \{ \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \ldots \}$$



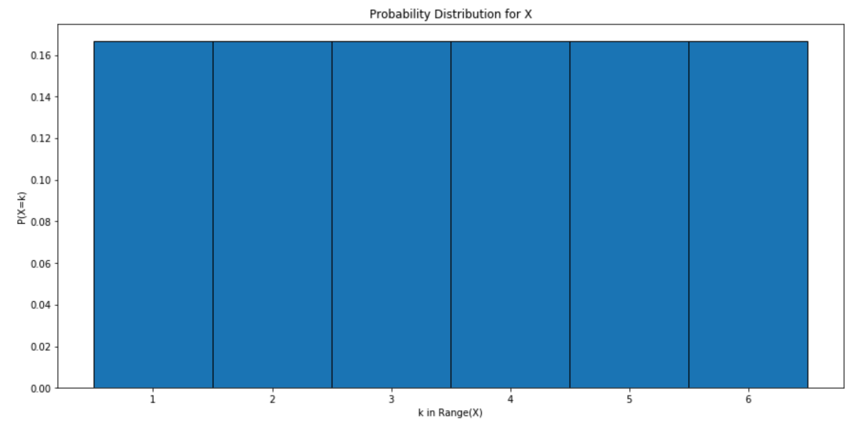
Functions of Discrete Random Variables

New random variables can be created by functions or expressions involving old random variables. But you have to be careful!

Example: X = "the number of dots on a thrown die"

$$R_X = \{1, 2, 3, 4, 5, 6\}$$

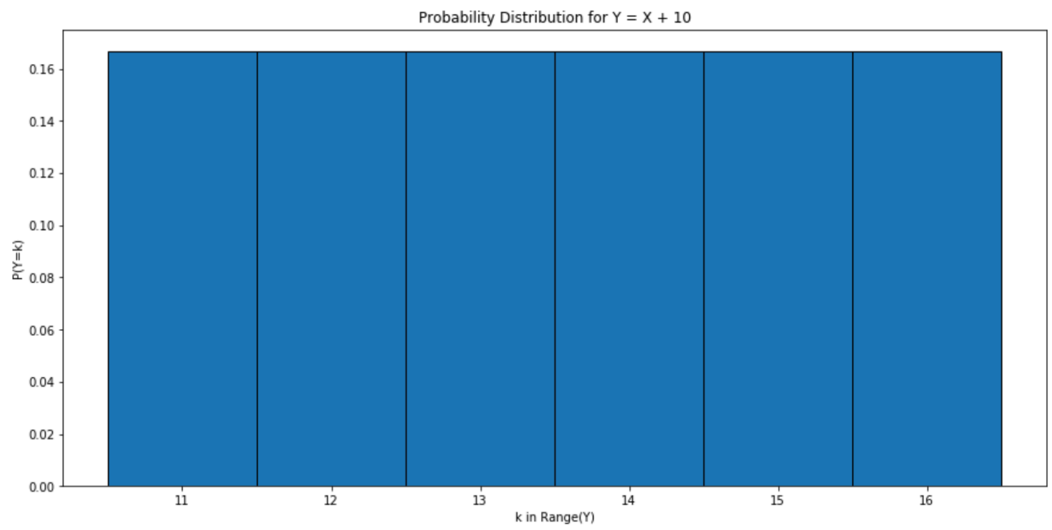
$$f_X = \{ \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6} \}$$



Let Y = X + 10

$$R_Y = \{ 11, 12, 13, 14, 15, 16 \}$$

$$f_Y = \{ \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6} \}$$



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Let Y = 2 * X

$$R_Y = \{\, 2, 4, 6, 8, 10, 12 \,\}$$

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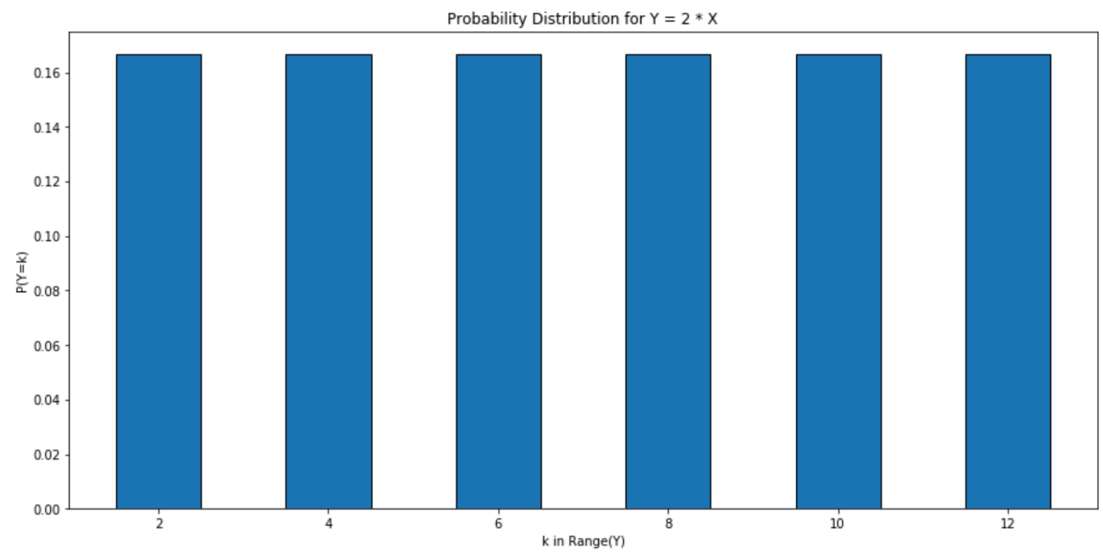

Let Y = 2^X

$$R_Y = \{\, 2, 4, 8, 16, 32, 64 \,\}$$

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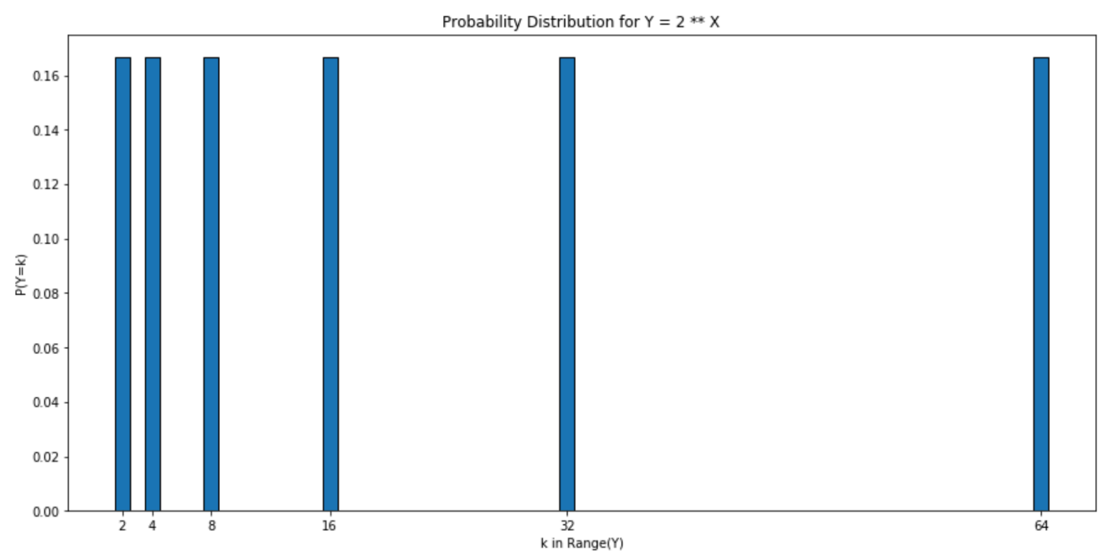

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One, the function of a random variable may combine outcomes...

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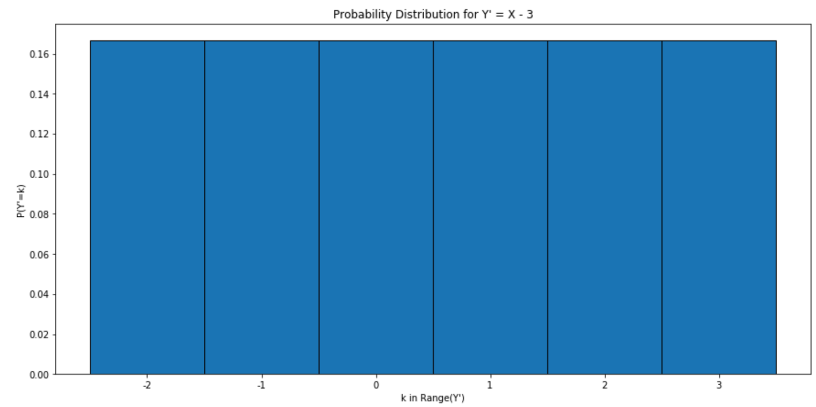

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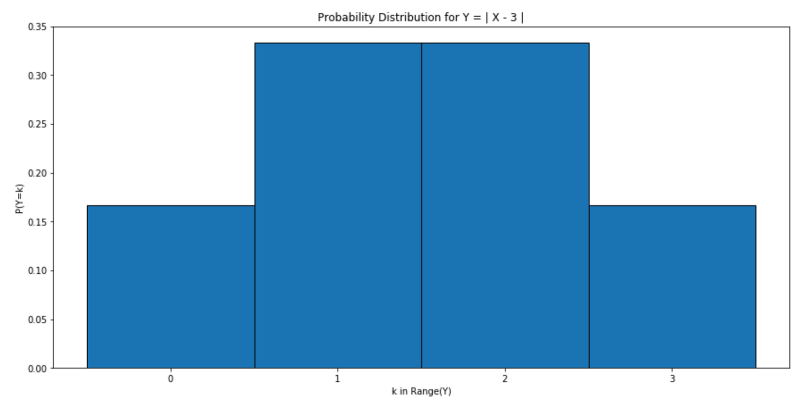

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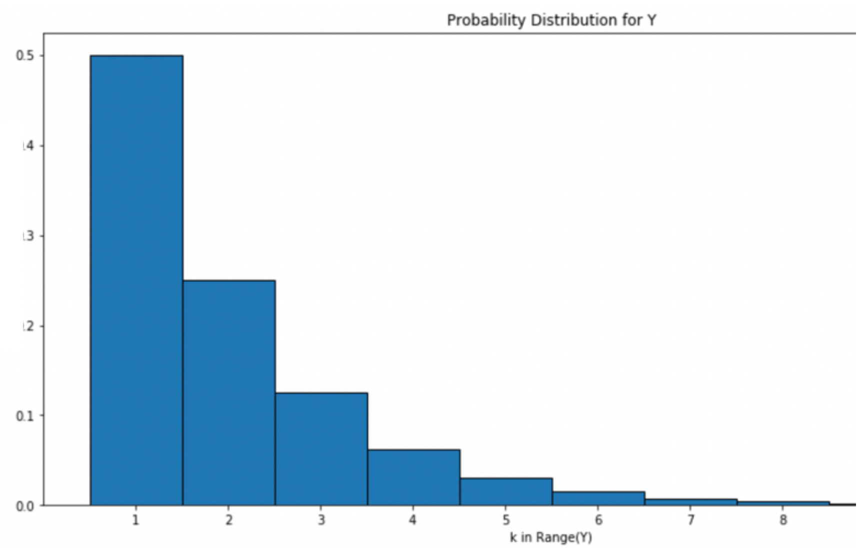

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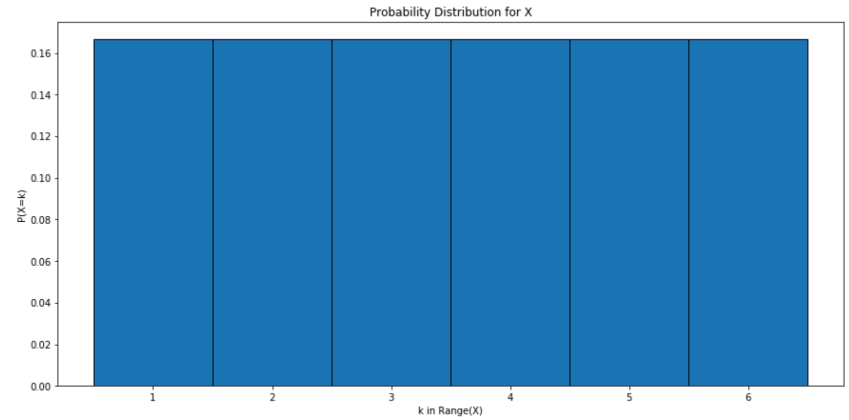

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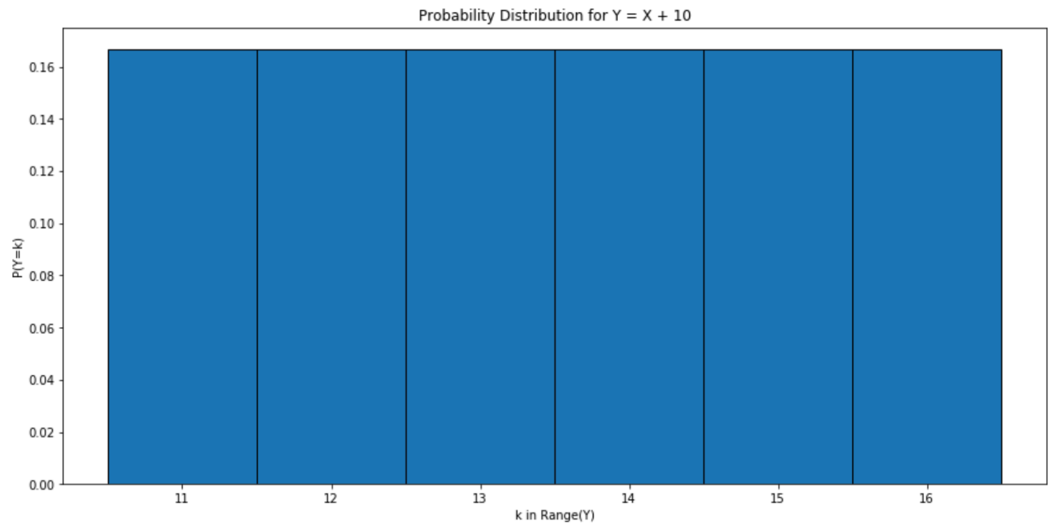

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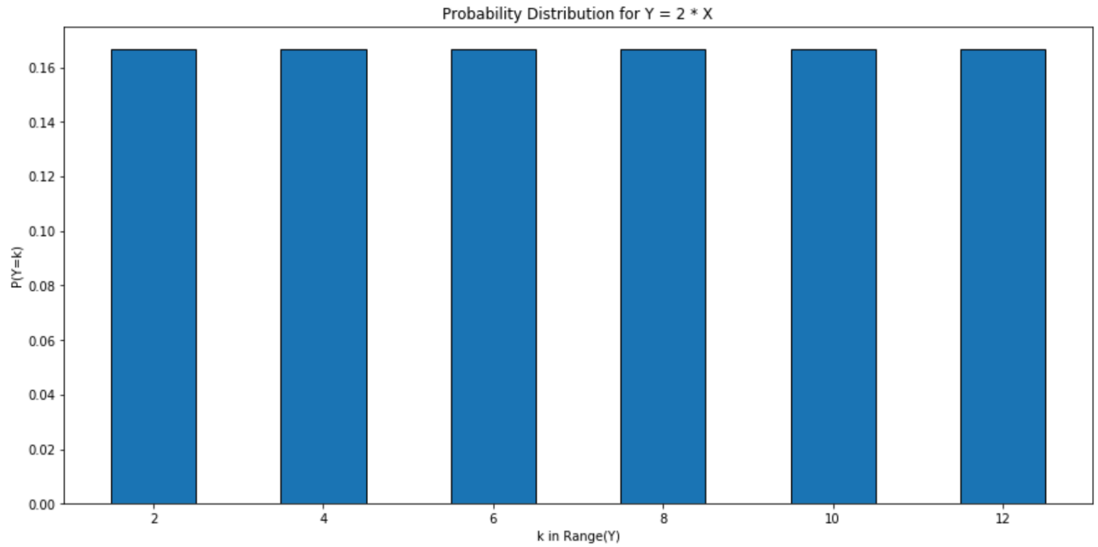

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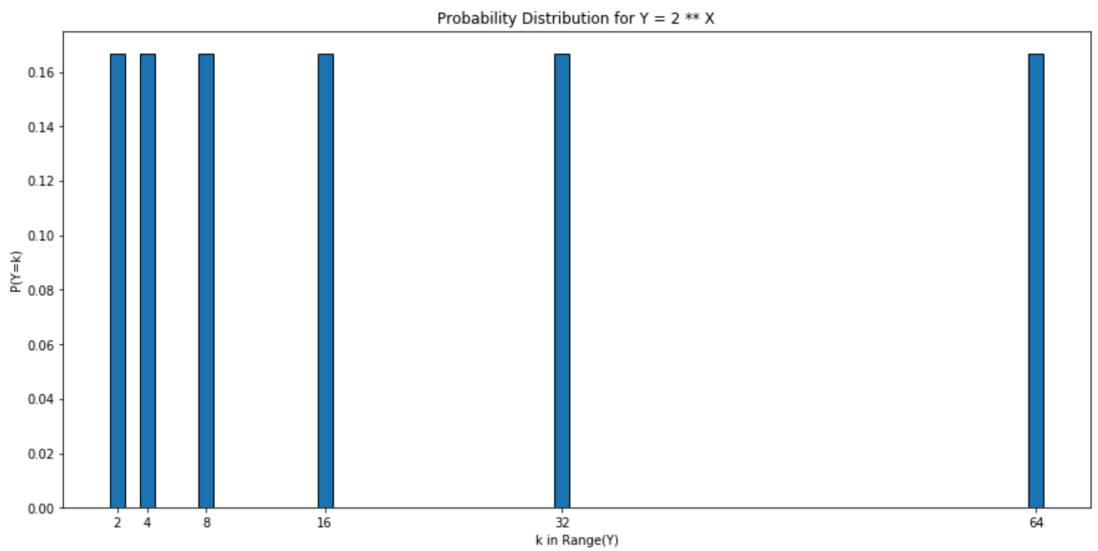

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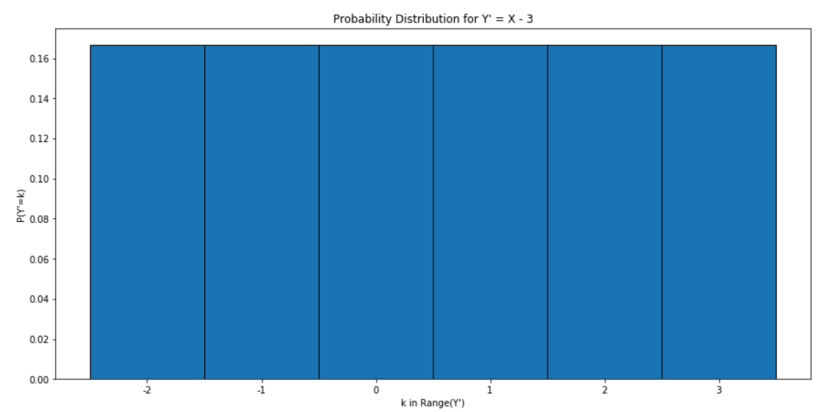

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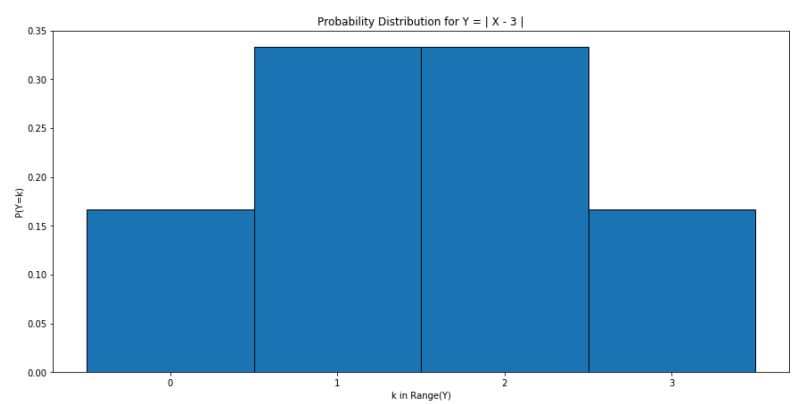


Functions of Discrete Random Variables

$$R_X = \{1, 2, 3, 4, 5, 6\}$$

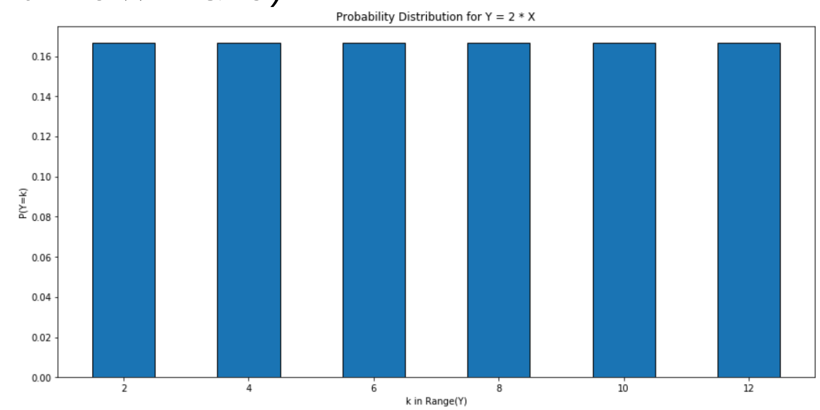
$$f_X = \{ \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6} \}$$

Two, you have to be careful when a random variable is used more than once, since each occurrence refers to a potentially different random outcome!

Let Y = 2 * X (twice the dots showing on a thrown die)

$$R_Y = \{ 2, 4, 6, 8, 10, 12 \}$$

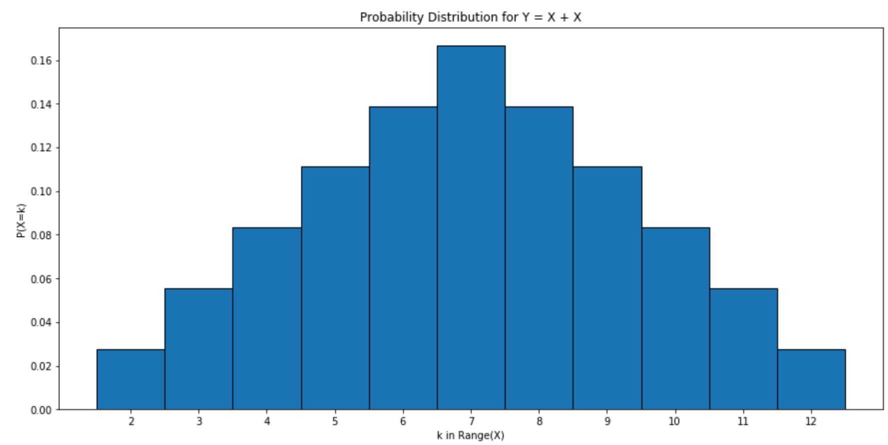
$$f_Y = \{ \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6}, \tfrac{1}{6} \}$$



Let Y = X + X (sum of the dots showing on two thrown dice)

$$R_Y = \{ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \}$$

$$f_Y = \{ \tfrac{1}{36}, \tfrac{2}{36}, \tfrac{3}{36}, \tfrac{4}{36}, \tfrac{5}{36}, \tfrac{6}{36}, \tfrac{5}{36}, \tfrac{4}{36}, \tfrac{3}{36}, \tfrac{2}{36}, \tfrac{1}{36} \}$$



Independence and Conditional RandomVariables

We will now use random variables to describe probability experiments, and all the ideas we have studied up to this point will be cast in terms of random variables. **Instead of events, we have RVs returning real numbers.** There is not much new to learn except to use the new notation.

Independence

Two random variables X and Y are **independent** iff and only if

$$\forall x, y \in R_X,\ \ P(X = j, Y = k) \ = \ P(X = j) \cdot P(Y = k)$$

$$P(A \cap B) = P(A) * P(B)$$

Conditional Random Variables

$$P(\, X = j \,|\, Y = k \,) \ = \ \frac{P(X = j, Y = k)}{P(Y = k)}$$

$$P(A \,|\, B) = \frac{P(A \cap B)}{P(B)}$$